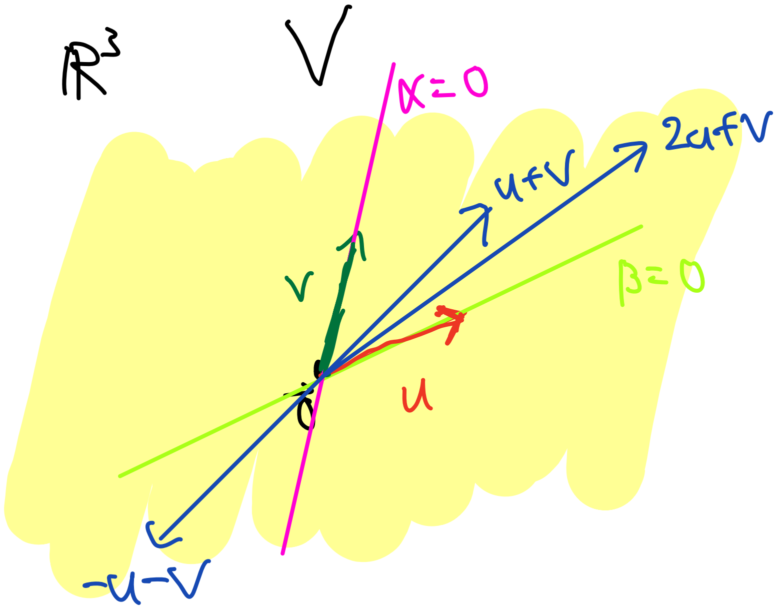
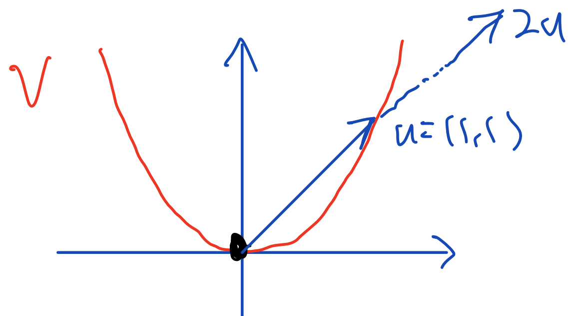
Definition 1.3.18 (Vector subspace). *We say that a subset* $V \subset \mathbb{R}^m$ *is a* **vector subspace** *of* $\mathbb{R}^m$ *if* V *contains the zero vector* $\mathbf{0}$ *and for any* $\mathbf{u}, \mathbf{v} \in V$, $\alpha, \beta \in \mathbb{R}$, *we have*

$$\alpha \mathbf{u} + \beta \mathbf{v} \in V.$$



eg Vector subspaces of $\mathbb{R}^2$?

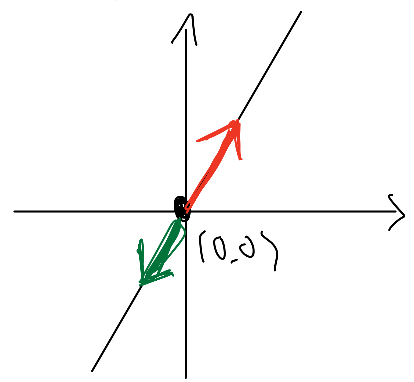
$\{(x,y): y = x^2\}$



$u \in V$, $2u \notin V$

Not vector subspace

$\{(x,y): 2x - y = 0\}$



$\alpha u + \beta v \in V$

Is Vector Subspace

Rmk Vector subspace in $\mathbb{R}^2$: $\{0\}$, lines through origin, $\mathbb{R}^2$

Definition 1.3.18 (Vector subspace). *We say that a subset $V \subset \mathbb{R}^m$ is a* **vector subspace** *of $\mathbb{R}^m$ if V contains the zero vector $\mathbf{0}$ and for any $\mathbf{u}, \mathbf{v} \in V$, $\alpha, \beta \in \mathbb{R}$, we have*

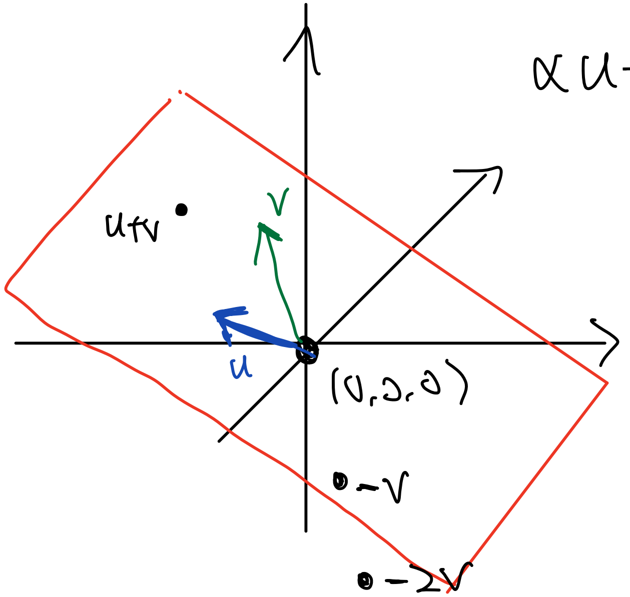
$$\underbrace{\alpha \mathbf{u} + \beta \mathbf{v}}_{\text{linear combinations of } u, v} \in V.$$

eg Vector subspaces of $\mathbb{R}^3$?

$\{(x,y,z) : x+y+z = 0\}$

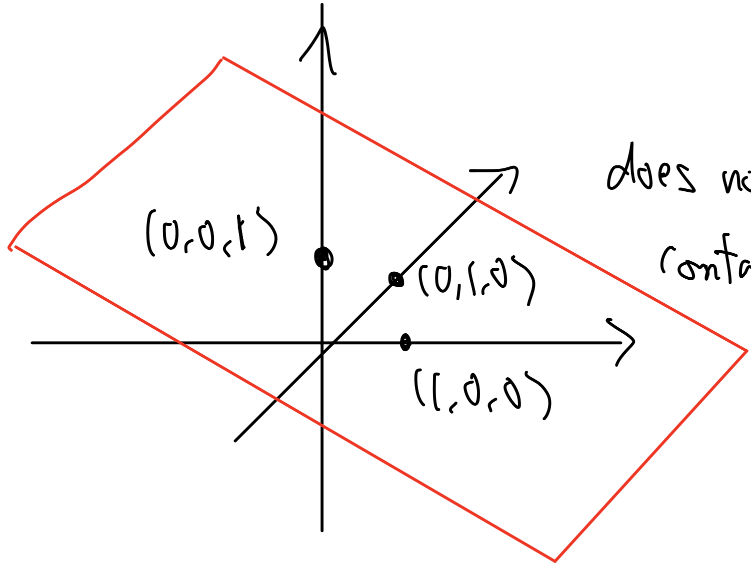
$\{(x,y,z) : x+y+z = 1\}$

$\alpha u + \beta v$



is a vector subspace

does not contain (0,0,0)



not a vector subspace

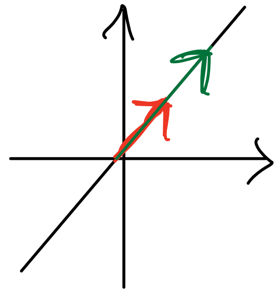
Rmk Vector subspace in $\mathbb{R}^3$: $\{\vec{0}\}$, line, plane through origin, $\mathbb{R}^3$

Definition 1.3.19 (Linearly independent vectors and spanning set). *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subset V$ be a set of vectors in V.*

1. *We say that E is* **linearly independent** *if*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

implies $c_1 = c_2 = \cdots = c_k = 0$.



<u>eg1</u> $\overset{v_1}{(1,1)}, \overset{v_2}{(1,2)}$

?

<u>eg 2</u> $(1,1), (2,2)$

If $c_1(1,1) + c_2(1,2) = (0,0)$

$$(c_1, c_1) + (c_2, 2c_2) = (0,0)$$

$$(c_1 + c_2, c_1 + 2c_2) = (0,0)$$

$$\begin{cases} c_1 + c_2 = 0 & \cdots (1) \\ c_1 + 2c_2 = 0 & \cdots (2) \end{cases}$$

(2) − (1) : $c_2 = 0 \cdots (3)$

Sub (3) into (1) $c_1 + 0 = 0$

$$c_1 = 0$$

linearly independent

If $c_1(1,1) + c_2(2,2) = (0,0)$

$$(c_1, c_1) + (2c_2, 2c_2) = (0,0)$$

$$(c_1 + 2c_2, c_1 + 2c_2) = (0,0)$$

$$\begin{cases} c_1 + 2c_2 = 0 \\ c_1 + 2c_2 = 0 \end{cases}$$

Many solutions

eg. $c_1 = 2$, $c_2 = -1$ non-zero solution

$$\not\Rightarrow \quad c_1 = c_2 = 0$$

linearly dependent

Definition 1.3.19 (Linearly independent vectors and spanning set). *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subset V$ be a set of vectors in V.*

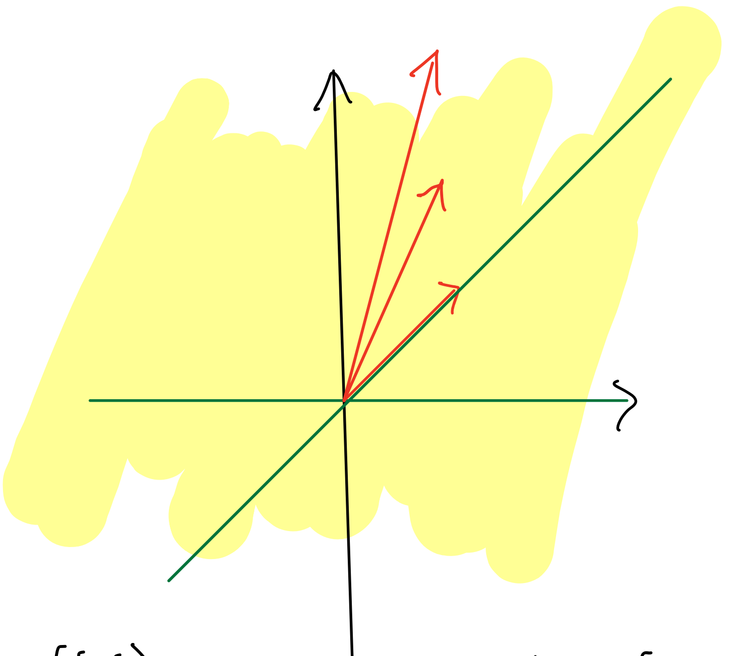
1. *We say that E is* **linearly independent** *if*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

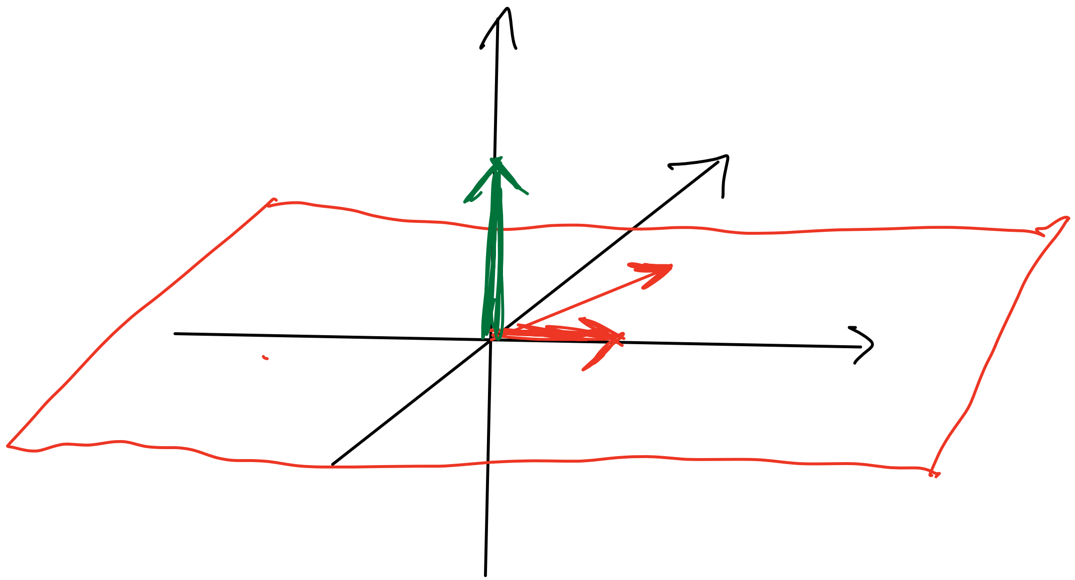
eg 3 $(1,1), (1,2), (1,3)$

eg 4 $(1,0,0), (1,1,0), (0,0,2)$ $\mathbb{R}^3$



$(1,1) - 2(1,2) + (1,3) = (0,0)$

linealy dependent



linearly independent

Definition 1.3.19 (Linearly independent vectors and spanning set). *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subset V$ be a set of vectors in V.*

1. *We say that E is* **linearly independent** *if*

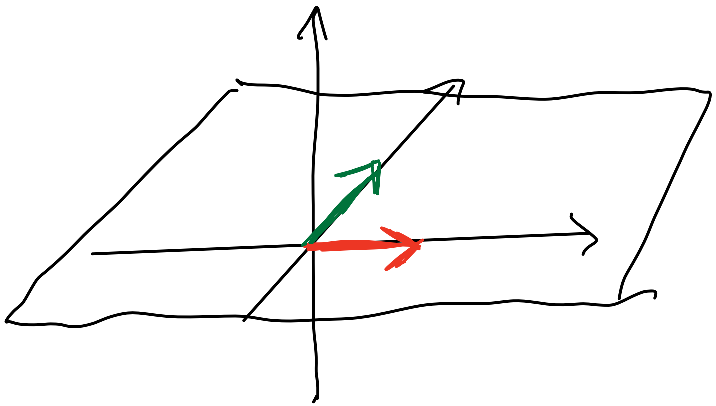
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

   implies $c_1 = c_2 = \cdots = c_k = 0$.

2. *We say that E* **spans** *V if for any $\mathbf{v} \in V$, there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ such that*

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k.$$

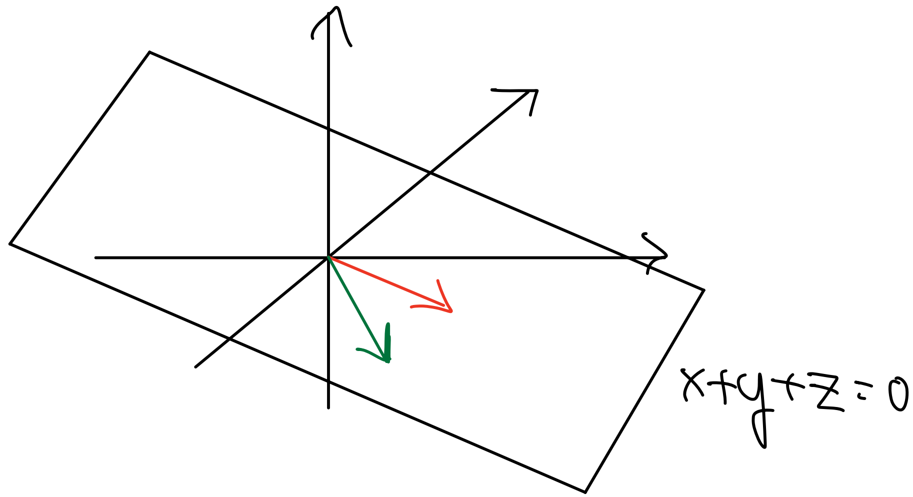
eg $E = \{\underline{(1,0,0)}, \underline{(0,1,0)}\}$



E spans $\{(x,y,z) : z = 0\}$

xy-plane

$E = \{(1,-1,0), (1,0,-1)\}$



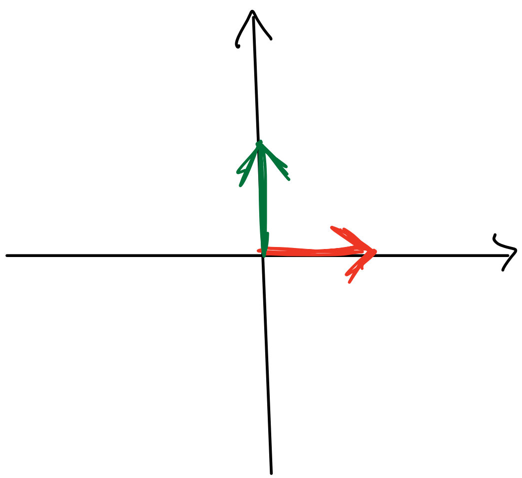
E spans $\{(x,y,z) : x+y+z=0\}$

Definition 1.3.23 (Basis). *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ be a set of vectors in V. We say that E constitutes a* **basis** *for V if*

1. *E is linearly independent, and*

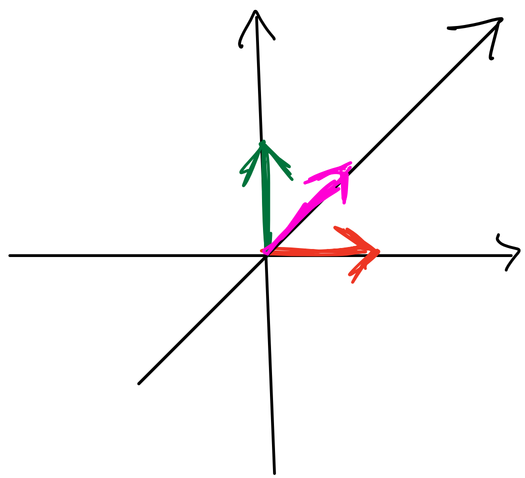
2. *E spans V.*

example: $e_1 = (1,0)$
$e_2 = (0,1)$



$\{e_1, e_2\}$ is a basis of $\mathbb{R}^2$

example: $e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$



$\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$

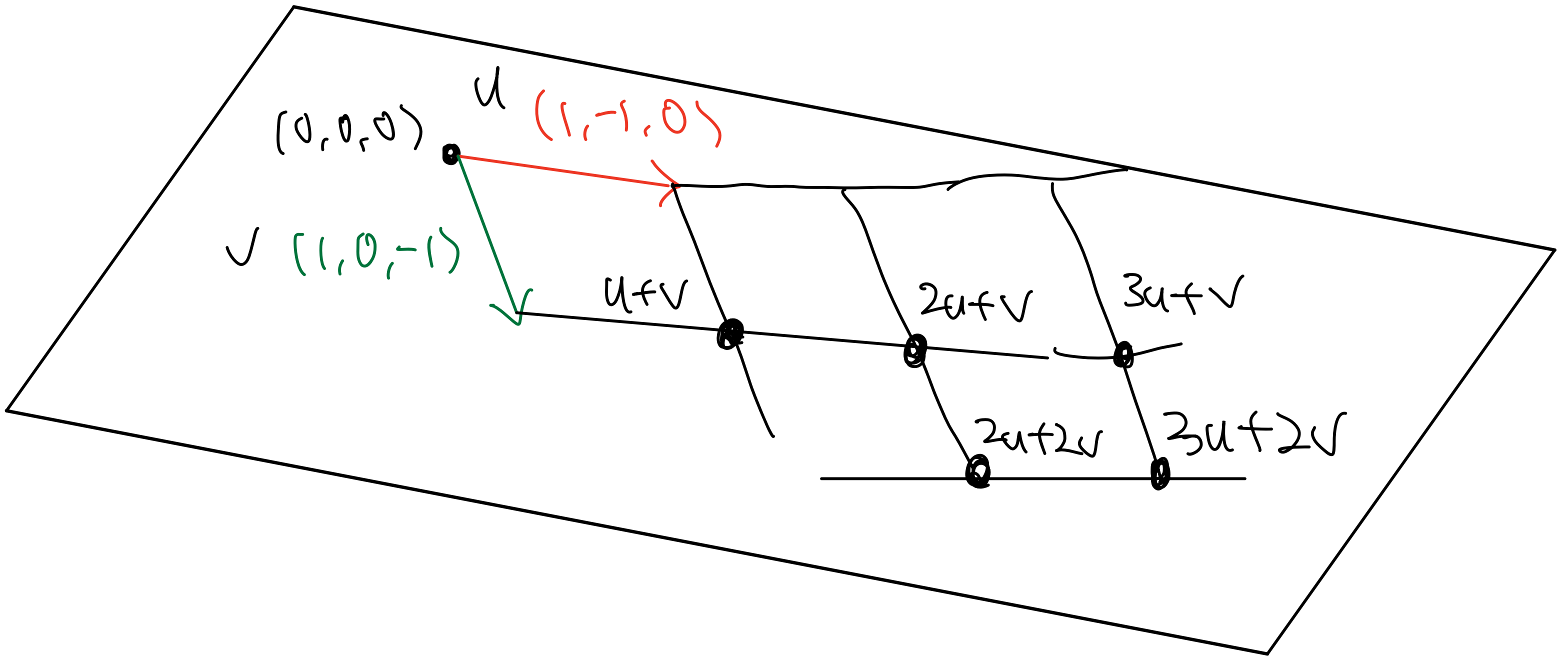
Definition 1.3.23 (Basis). *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ be a set of vectors in V. We say that E constitutes a* **basis** *for V if*

1. *E is linearly independent, and*

2. *E spans V.*

example : $V = \{(x,y,z) : x+y+z=0\}$

$\{\alpha u + \beta v : \alpha, \beta \in \mathbb{R}\}$



$E = \{(1,-1,0), (1,0,-1)\}$ is a basis of V

Theorem 1.3.25. *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ be a set of vectors in V. Then the following conditions are equivalent.*

1. *E constitutes a basis for V.*

2. *For any $\mathbf{v} \in V$, there exists unique $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that*

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n.$$

Definition 1.3.28 (Dimension)**.** *Let V be a vector subspace of $\mathbb{R}^m$. The* **dimension** *of V is the number of vectors in a basis for V and is denoted by* $\dim(V)$.

Theorem 1.3.33. *The following conditions for $n \times n$ matrix A are equivalent.*

1. $\det(A) \neq 0$
2. *A is invertible, that is, the inverse A^{-1} of A exists.*
3. *For any n column vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for $\mathbf{x}$.*
4. *The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has no nontrivial solution, that is, solution for which $\mathbf{x} \neq \mathbf{0}$.*
5. *The column vectors of A constitute a basis for $\mathbb{R}^m$.*

$$\begin{cases} (1)x_1 + 2x_2 = 0 \\ 3x_1 + 4x_2 = 0 \end{cases}$$

eg 1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det A = -2 \neq 0$ A is invertible $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

has only zero solution

eg 2

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\det A = 0$ A is not invertible $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

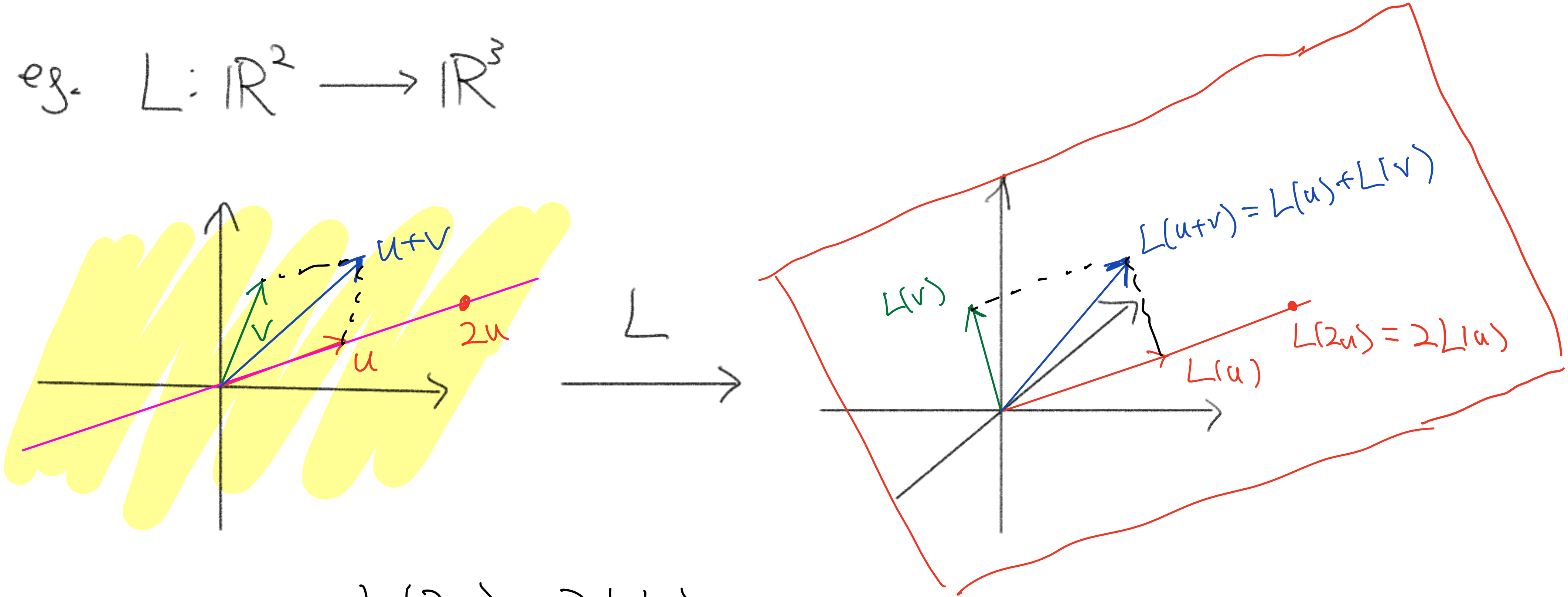
has non-trivial solution eg $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

1.4 Orthogonal matrices and rigid transformations

Definition 1.4.1 (Linear transformation). *A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is called a* **linear transformation** *if for any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$, *then*

$$L(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha L(\mathbf{u}) + \beta L(\mathbf{v})$$

e.g. $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$



$L(2u) = 2L(u)$

$L(u+v) = L(u) + L(v)$

Example 1.4.2 (Linear transformations associated with matrices). *Let A be an $\underline{m \times n \text{ matrix}}$. Define a function $L_A : \mathbb{R}^n \to \mathbb{R}^m$ by*

$$\underset{A}{} \qquad L_A(\mathbf{v}) = A\mathbf{v} \qquad \underset{m\times n}{A} \quad \underset{n\times 1}{v} \quad \underset{m\times 1}{Av}$$

for $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$. Here we use the column vector notation where $\mathbf{v}$ is an n column vector and $A\mathbf{v}$ is an m column vector. Then L_A is a linear transformation which is called the linear transformation associated with A.

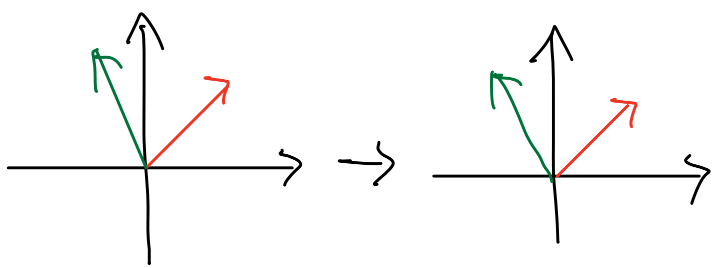
<u>eg</u> $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

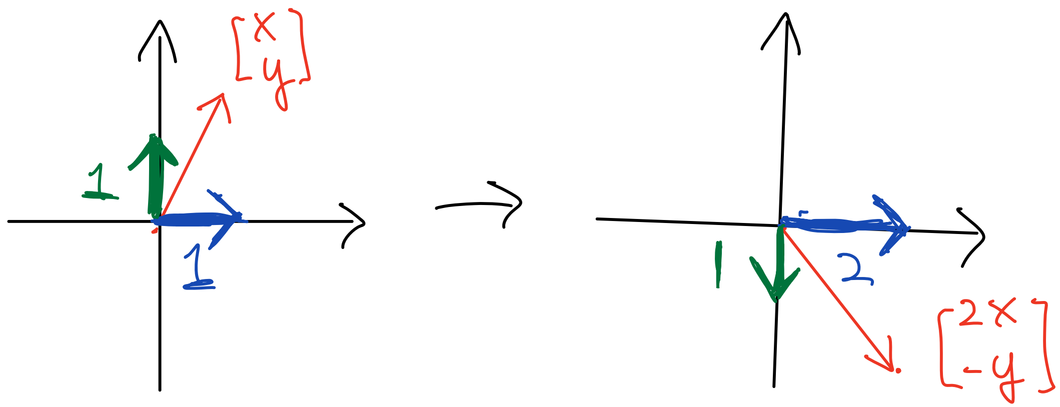
$$L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

<u>eg</u> $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$$L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix}$$

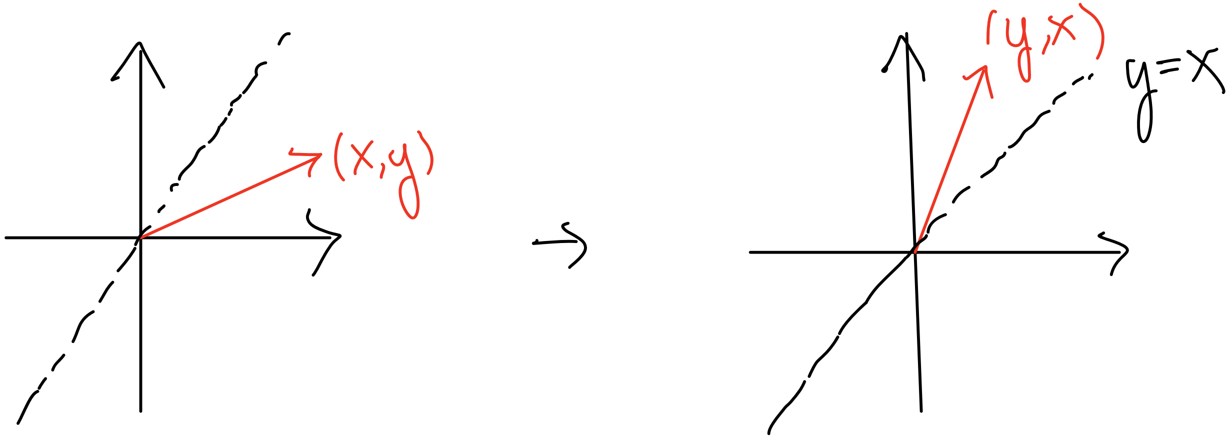
Identity transformation





eg $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



Reflection across the line $y = x$

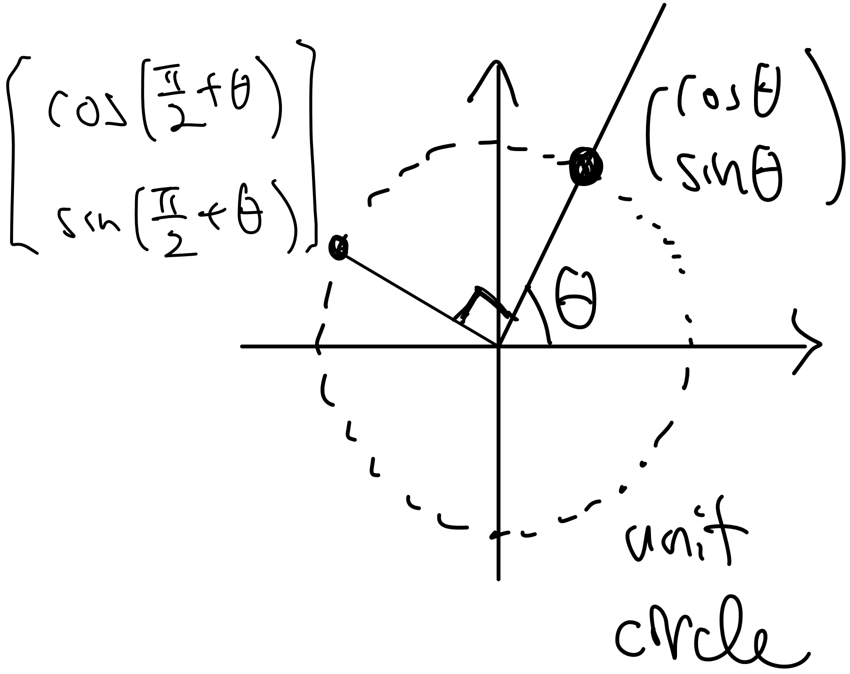
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

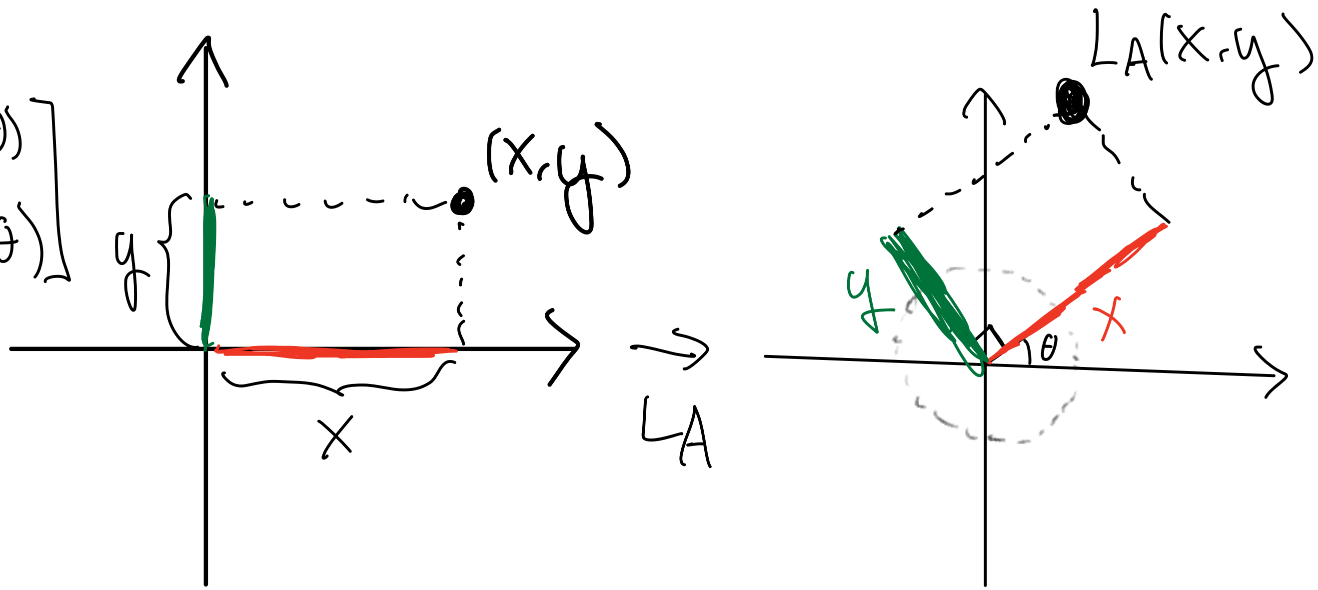
$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$$= x\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + y\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= x\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + y\begin{bmatrix} \cos(\frac{\pi}{2}+\theta) \\ \sin(\frac{\pi}{2}+\theta) \end{bmatrix}$$



Rotation by θ anti-clockwisely



Proposition 1.4.3 (Matrix representation of linear transformation). *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is an $m \times n$ matrix A such that $L_A = L$ where L_A is the linearly transformation associated with A. The matrix A is called the* **matrix representation** *of L.*

$$\left\{ \begin{array}{c} \text{Linear transformation} \\ \mathbb{R}^n \to \mathbb{R}^m \end{array} \right\} \longleftrightarrow \left\{ m \times n \text{ matrix} \right\}$$

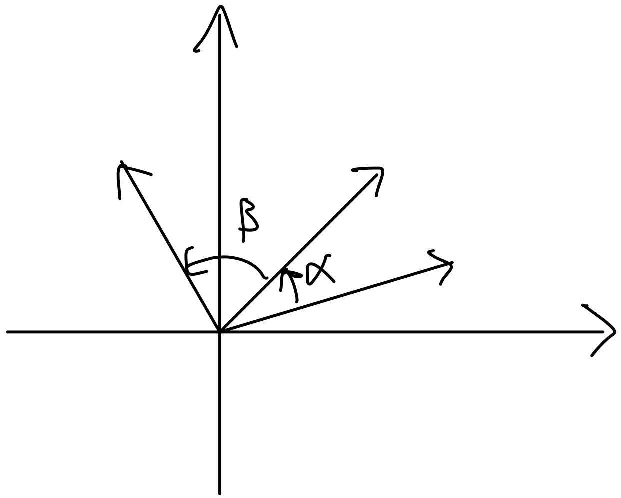
$$\mathbb{R}^n \xrightarrow{L_B} \mathbb{R}^m \xrightarrow{L_A} \mathbb{R}^k \qquad \begin{array}{cc} A & B \\ k \times m & m \times n \end{array}$$

$$L_A \circ L_B = L_{AB} \qquad \underline{\text{Rmk}}\ (L_A \circ L_B)(\vec{v}) = L_A(L_B(\vec{v}))$$

Proposition 1.4.4. *Let A and an $k \times m$ matrix and B be an $m \times n$ matrix. Let L_A and L_B be the linear transformation associated with A and B respectively. Then the matrix representing $L_A \circ L_B$ is AB. In other words,*

$$L_{AB} = L_A \circ L_B.$$

$$\underbrace{\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\underbrace{\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}_{\text{Rotate } \begin{bmatrix}x\\ y\end{bmatrix} \text{ by } \alpha}}_{\text{Rotate further by } \beta}$$

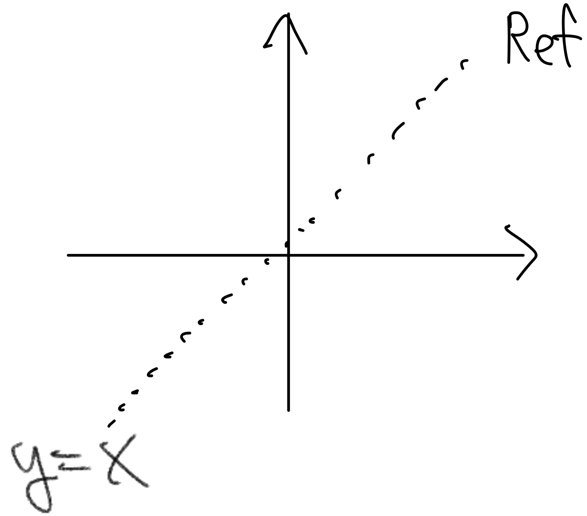


Rotate by $\alpha+\beta$ totally

$$\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}\cos\beta\cos\alpha-\sin\beta\sin\alpha & -\cos\beta\sin\alpha-\sin\beta\cos\alpha\\ \sin\beta\cos\alpha+\cos\beta\sin\alpha & -\sin\beta\sin\alpha+\cos\beta\cos\alpha\end{bmatrix}$$

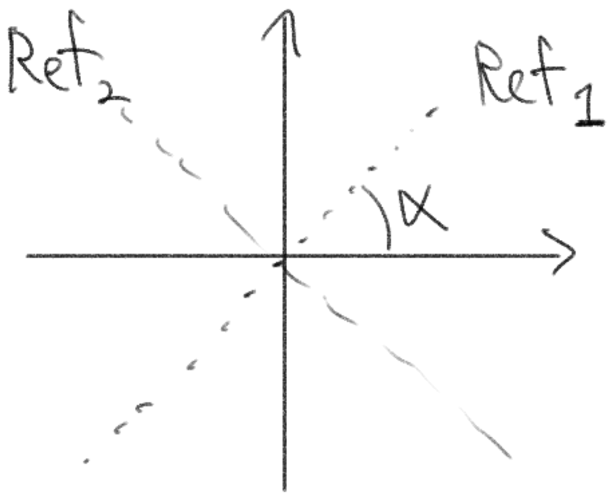
$$= \begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$$

Q: What are the compositions?



then rotate by θ

$\boxed{12:20}$



$\text{Ref}_2 \circ \text{Ref}_1 = ?$

$$\text{Ref}_1 \sim \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

1.5 Eigenvalues, eigenvectors and diagonalization

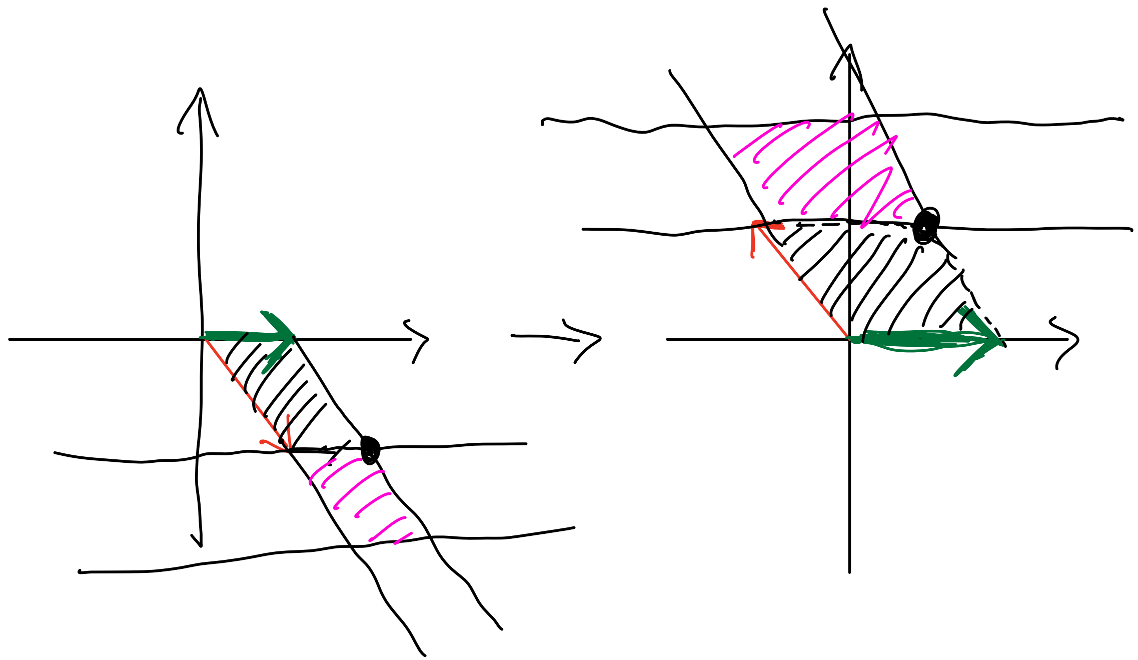
Definition 1.5.1 (Eigenvalues and eigenvectors)**.** *Let A be an $n \times n$ matrix. If λ is a complex number*[3] *and $\boldsymbol{\xi}$ is a non-zero*[4] *complex vector such that*

$$A\boldsymbol{\xi} = \lambda\boldsymbol{\xi},$$

then we say that λ is an **eigenvalue** *of A and $\boldsymbol{\xi}$ is an* **eigenvector** *of A associated with λ.*

$$\begin{bmatrix}2 & 3\\ 0 & -1\end{bmatrix}\overset{u}{\begin{bmatrix}1\\ -1\end{bmatrix}} = \begin{bmatrix}-1\\ 1\end{bmatrix} = (-1)\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\begin{bmatrix}2 & 3\\ 0 & -1\end{bmatrix}\underset{v}{\begin{bmatrix}1\\ 0\end{bmatrix}} = \begin{bmatrix}2\\ 0\end{bmatrix} = 2\begin{bmatrix}1\\ 0\end{bmatrix}$$



u, v Eigenvector ($\neq \vec{0}$)

$-1, 2$ eigenvalue

Definition 1.5.2 (Characteristic polynomial and characteristic equation). *Let A be an $n \times n$ matrix. The* **characteristic polynomial** *of A is the degree n polynomial* $\det(xI - A)$ *in x, where I is the identity matrix. The* **characteristic equation** *of A is the degree n polynomial equation*

$$\det(xI - A) = 0.$$

Discussed later

1.7 Some transcendental functions

1. **Exponential function**:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \text{ for } x \in \mathbb{R}$$

2. **Trigonometric functions**: *There are 6 trigonometric functions which are defined as follows.*

Cosine: $$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \text{ for } x \in \mathbb{R}$$

Sine: $$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for } x \in \mathbb{R}$$

Tangent: $$\tan x = \frac{\sin x}{\cos x} \text{ for } x \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$$

Cotangent: $$\cot x = \frac{\cos x}{\sin x} \text{ for } x \neq k\pi,\ k \in \mathbb{Z}$$

Secant: $$\sec x = \frac{1}{\cos x} \text{ for } x \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$$

Cosecant: $$\csc x = \frac{1}{\sin x} \text{ for } x \neq k\pi,\ k \in \mathbb{Z}$$

3. **Hyperbolic functions**: *There are 6 hyperbolic functions which are defined as follows.*
Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \text{ for } x \in \mathbb{R}$$

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \text{ for } x \in \mathbb{R}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} \text{ for } x \in \mathbb{R}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} \text{ for } x \neq 0$$

Hyperbolic secant:

$$\mathrm{sech} x = \frac{1}{\cosh x} \text{ for } x \in \mathbb{R}$$

Hyperbolic cosecant:

$$\mathrm{csch} x = \frac{1}{\sinh x} \text{ for } x \neq 0$$

Theorem 1.7.2. *The exponential function satisfies*

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

for any $x \in \mathbb{R}$.

Definition 1.7.3 (Logarithmic function)**.** *The* **logarithmic function** *is the function $\ln : \mathbb{R}^+ \to \mathbb{R}$ defined for $x > 0$ by*

$$y = \ln x \text{ if } e^y = x.$$

In other words, $\ln x$ is the inverse function of the exponential function.

Proposition 1.7.4 (Identities for transcendental functions).

1. *Exponential function:*
   (a) $e^{x+y} = e^x e^y$
   (b) $e^{x-y} = \dfrac{e^x}{e^y}$
   (c) $e^{kx} = (e^x)^k$ *for* $k \in \mathbb{Z}$

2. *Logarithmic function:*
   (a) $\ln(xy) = \ln x + \ln y$
   (b) $\ln \dfrac{x}{y} = \ln x - \ln y$
   (c) $\ln(x^k) = k \ln x$ *for* $k \in \mathbb{Z}$

3. *Trigonometric identities:*
   (a) $\cos^2 x + \sin^2 x = 1; \quad \sec^2 x - \tan^2 x = 1; \quad \csc^2 x - \cot^2 x = 1$
   (b) $\cos(-x) = \cos x; \quad \sin(-x) = -\sin x; \quad \tan(-x) = -\tan x$
   (c) $\cos(x+y) = \cos x \cos y - \sin x \sin y;$
   $\sin(x+y) = \sin x \cos y + \cos x \sin y;$
   $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
   (d) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x;$
   $\sin 2x = 2 \sin x \cos x;$
   $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

4. *Hyperbolic identities:*
   (a) $\cosh^2 x - \sinh^2 x = 1; \quad \mathrm{sech}^2 x + \tanh^2 x = 1; \quad \coth^2 x - \mathrm{csch}^2 x = 1$
   (b) $\cosh(-x) = \cosh x; \quad \sinh(-x) = -\sinh x; \quad \tanh(-x) = -\tanh x$
   (c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y;$
   $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y;$
   $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
   (d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x;$
   $\sinh 2x = 2 \sinh x \cosh x;$
   $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Proposition 1.7.5 (Derivatives of transcendental functions).

1. *Exponential function:*
$$\frac{d}{dx}e^x = e^x$$

2. *Logarithmic function:*
$$\frac{d}{dx}\ln x = \frac{1}{x}$$

3. *Trigonometric functions:*
$$\frac{d}{dx}\cos x = -\sin x; \quad \frac{d}{dx}\sin x = \cos x;$$
$$\frac{d}{dx}\tan x = \sec^2 x; \quad \frac{d}{dx}\cot x = -\csc^2 x;$$
$$\frac{d}{dx}\sec x = \sec x\tan x; \quad \frac{d}{dx}\csc x = -\csc x\cot x$$

4. *Inverse trigonometric functions*[7]*:*
$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}};$$
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}};$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

5. *Hyperbolic functions:*
$$\frac{d}{dx}\cosh x = \sinh x; \quad \frac{d}{dx}\sinh x = \cosh x;$$
$$\frac{d}{dx}\tanh x = \mathrm{sech}^2 x; \quad \frac{d}{dx}\coth x = -\mathrm{csch}^2 x;$$
$$\frac{d}{dx}\mathrm{sech} x = -\mathrm{sech} x\tanh x; \quad \frac{d}{dx}\mathrm{csch} x = -\mathrm{csch} x\coth x$$

6. *Inverse hyperbolic functions*[8]*:*
$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}};$$
$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}};$$
$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$$

Proposition 1.7.6 (Integrals of transcendental functions).

1. *Exponential function:*
$$\int e^x dx = e^x + C$$

2. *Logarithmic function:*
$$\int \frac{1}{x} dx = \ln|x| + C$$

3. *Trigonometric functions:*
$$\int \cos x dx = \sin x + C; \qquad \int \sin x dx = -\cos x + C;$$
$$\int \tan x dx = \ln \sec x; \qquad \int \cot x = \ln \sin x + C;$$
$$\int \sec x dx = \ln|\sec x + \tan x| + C; \quad \int \csc x dx = \ln|\csc x - \cot x| + C$$

4. *Hyperbolic functions:*
$$\int \cosh x dx = \sinh x + C; \qquad \int \sinh x dx = \cosh x + C;$$
$$\int \tanh x dx = \ln \cosh x; \qquad \int \coth x = \ln \sinh x + C;$$
$$\int \mathrm{sech} x dx = \tan^{-1} \sinh x + C; \quad \int \mathrm{csch} x dx = \ln|\mathrm{csch} x - \coth x| + C$$

2 Curves

2.1 Regular parametrized curves

Definition 2.1.1 (Regular parametrized curves). *A* **regular parametrized curve** *is a differentiable function* $\mathbf{r} : (a, b) \to \mathbb{R}^n$*,* $n = 2$ *or 3, such that* $\mathbf{r}'(t) \neq \mathbf{0}$ *for any* $t \in (a, b)$.

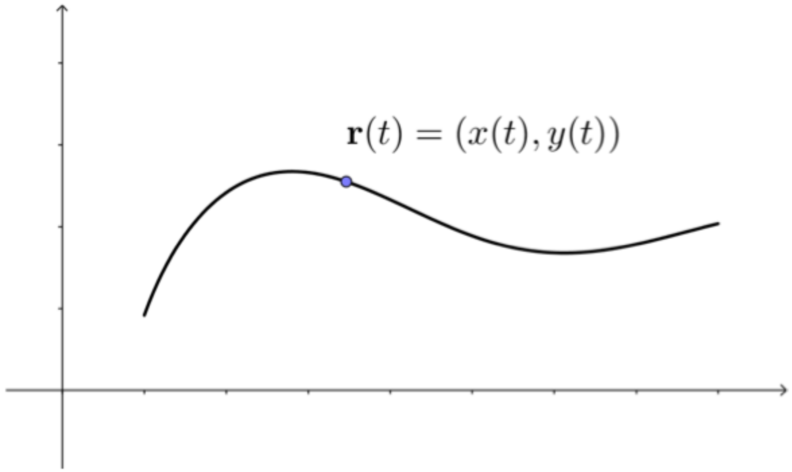
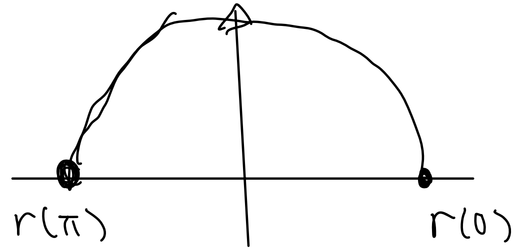
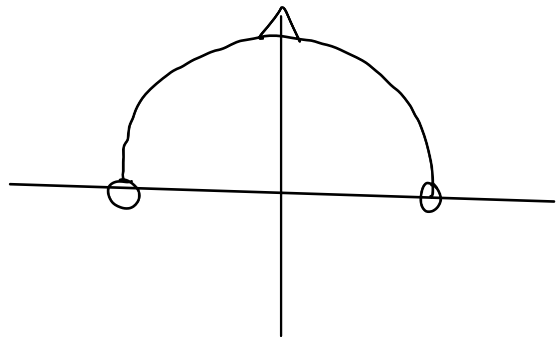


Figure 1: Regular parametrized curve

$r(t) = (2\cos t, 2\sin t), \quad 0 \leq t \leq \pi$



$\mathbf{r}(t) = (2\cos t, 2\sin t) \quad 0 < t < \pi$



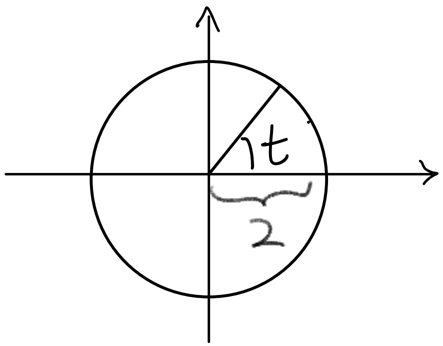
We will also consider curve defined on closed interval and unbounded interval

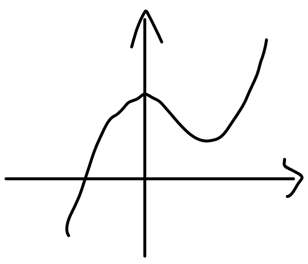
Parametrization

$x(t) = 2\cos t$

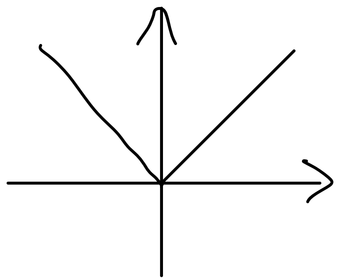
$y(t) = 2\sin t$

$t \in (-\infty, \infty)$





differentiable

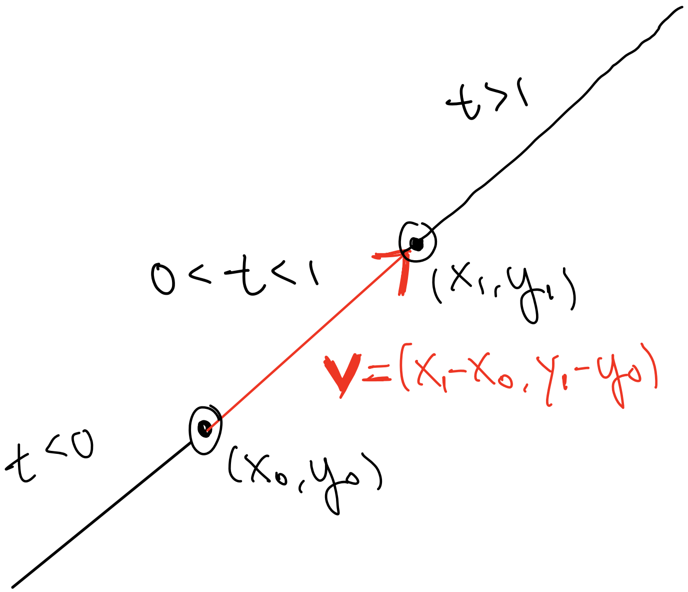


continuous, not differentiable

$t > 0$

1. *Straight line: Let (x_0, y_0) and (x_1, y_1) be two points on $\mathbb{R}^2$. The function*

$$\mathbf{r}(t) = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1), \ \textit{for } 0 < t < 1$$



$$\vec{r}(t) = (x_0, y_0) + t\mathbf{v}$$
$$= (x_0, y_0) + t(x_1 - x_0, y_1 - y_0)$$
$$= (x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0))$$
$$= ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1)$$

2. *Circle: Let $r > 0$ be a positive real number. The function*

$$\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta), \ \textit{for } 0 < \theta < 2\pi$$

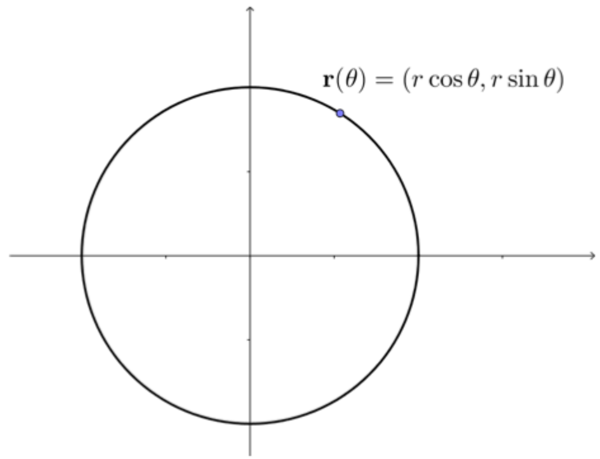


Figure 3: Circle

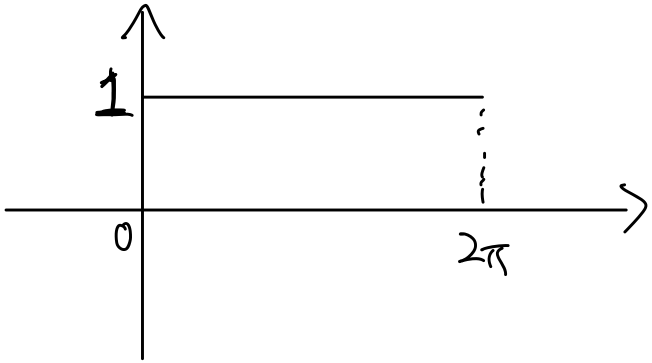
3. Cycloid: The function

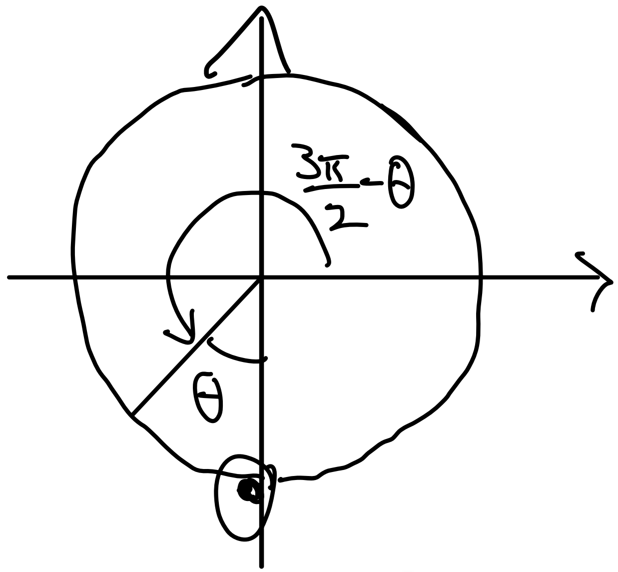
$$\mathbf{r}(\theta) = (\theta - \sin\theta, 1 - \cos\theta), \ \textit{for } 0 < \theta < 2\pi$$

$$= \underbrace{(\theta, 1)}_{r_1} + \underbrace{(-\sin\theta, -\cos\theta)}_{r_2}$$

$$(-\sin\theta, -\cos\theta)$$

$$= \left(\cos\left(\frac{3\pi}{2} - \theta\right), \sin\left(\frac{3\pi}{2} - \theta\right)\right)$$





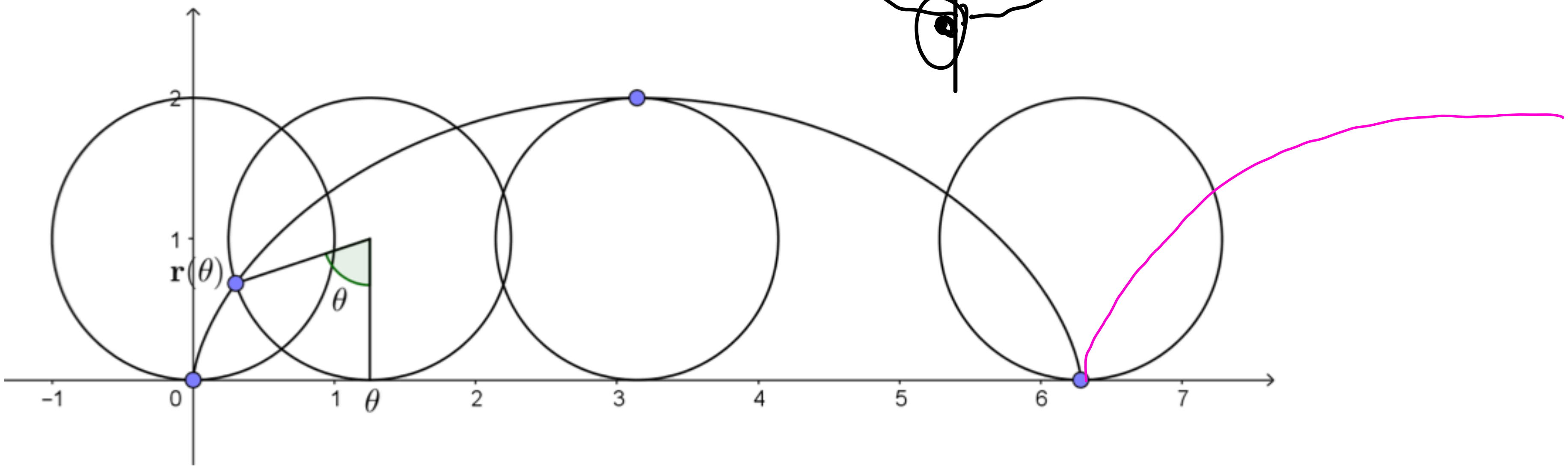


Figure 4: Cycloid

Note $r(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$ $\quad 0 \leq \theta < 4\pi$

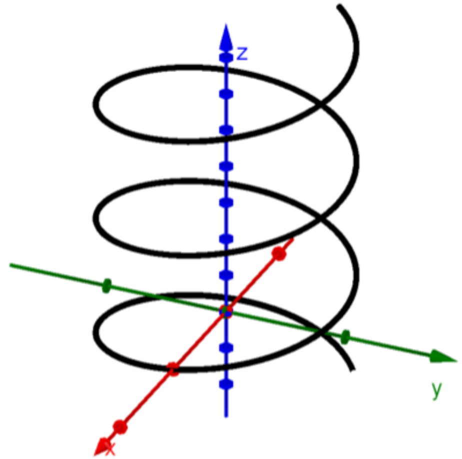
$r'(\theta) = (1 - \cos\theta, \sin\theta)$

$r'(2\pi) = (1-1, 0) = (0,0)$ ← not regular

4. Helix: The function $= (a\cos\theta, a\sin\theta, 0) + (0, 0, b\theta)$

$$\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, b\theta), \ \textit{for } \theta \in \mathbb{R}$$

defines a curve which is called a **helix***.*



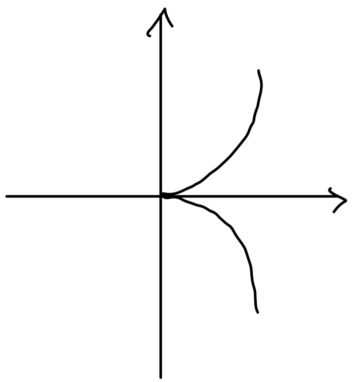
$\quad x \quad y$

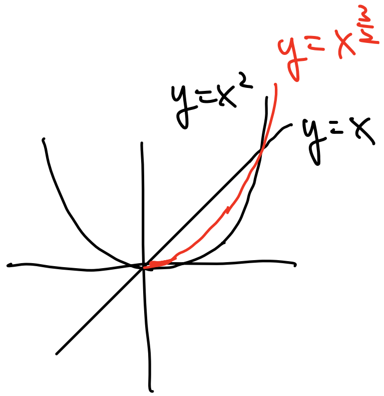
Example 2.1.3. *Let* $\mathbf{r}(t) = (t^2, t^3)$. *Then* $\mathbf{r}'(t) = (2t, 3t^2)$ *and* $\mathbf{r}'(0) = (0, 0)$. *Therefore* $\mathbf{r}(t)$ *is not regular at* $t = 0$.

$x = t^2$

$y = t^3$ $\Rightarrow$ $x^3 = y^2$

$y = \pm x^{\frac{3}{2}}$

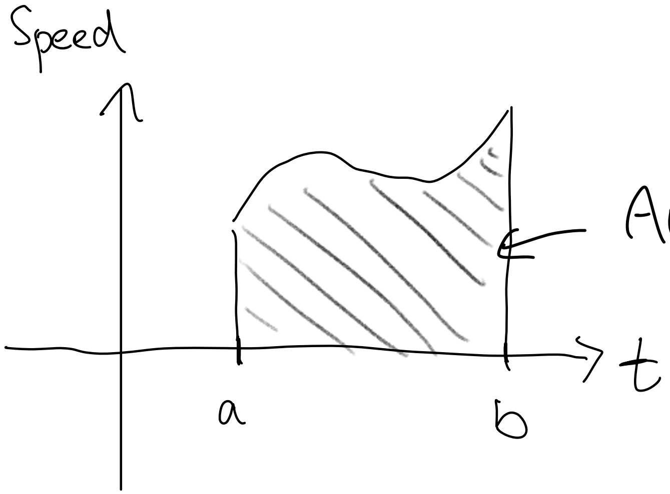




2.2 Arc length

Definition 2.2.1 (Arc length)**.** *Let* $\mathbf{r} : (a, b) \to \mathbb{R}^n$ *be a regular parametrized curve. Then the* **arc length** *of* $\mathbf{r}$ *is defined by*

$$l = \int_a^b \|\mathbf{r}'(t)\| dt.$$



Area = distance

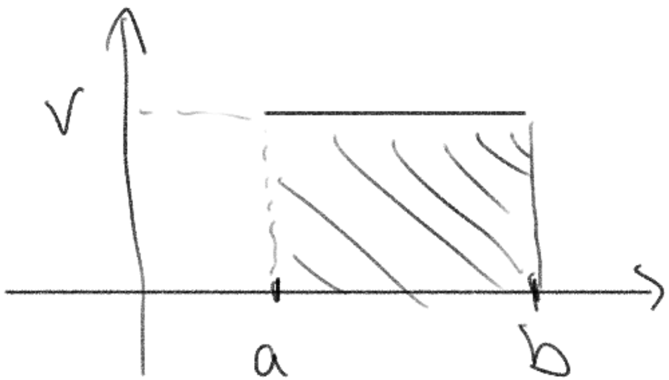
= arclength

$r(t)$ = location/displacent

$r'(t)$ = velocity

$\|r'(t)\|$ = speed

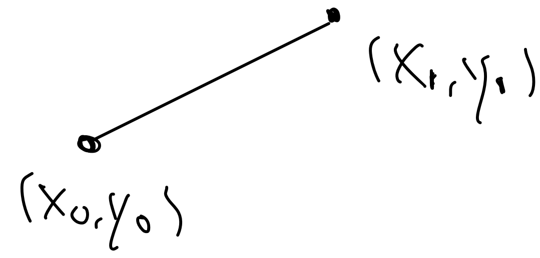
Constant speed:



distance = $v(b-a)$

Example 2.2.2 (Arc length of line segments). *Let*

$$\mathbf{r}(t) = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1), \ 0 < t < 1,$$



$$r'(t) = (-x_0 + x_1, -y_0 + y_1)$$

$$\|r'(t)\| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

$$l = \int_0^1 \|r'(t)\| dt = \int_0^1 \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}\, dt$$

$$= \left[\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}\, t\right]_0^1 = \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}$$

Example 2.2.3 (Arc length of circles). *Let* $\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta)$, $0 < \theta < 2\pi$, *be the circle with radius* $r > 0$ *centered at the origin. Now*

$$r(\theta) = (r\cos\theta, r\sin\theta)$$

$$r'(\theta) = (-r\sin\theta, r\cos\theta)$$

$$\|r'(\theta)\| = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2} = r$$

$$l = \int_0^{2\pi} \|r'(\theta)\| d\theta = \int_0^{2\pi} r\, d\theta = [r\theta]_0^{2\pi} = 2\pi r$$